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Rapports de Recherche

N° 487

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Février 1986

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Abstract- In this paper we consider a class of quasilinear parabolic systems in several spaces variables :

$$(*)\quad \begin{cases} \dfrac{\partial u}{\partial t} + \sum\limits_{i=1}^{N} B_i(t,x,u) \dfrac{\partial u}{\partial x_i} + C(t,x,u) = \mu \, \Delta u \ \text{ in } \pi_T \\[2ex] u(x,o) = u_o(x) \quad \text{ in } \mathbb{R}^N \end{cases}$$

Where B_i, C and u_o are given data , $\mu > o$ fixed and

$\pi_T =]o, T[\times \mathbb{R}^N$, $u = (u^1, u^2, \ldots, u^M)$.

Uning probabilistic methods we introduce a notion of generalized solution of (*) and we prove existence and uniqueness results.

Key words : quasilinear - generalized solution

Résumé- Dans ce papier on considère une classe de systemes quasilinéaires paraboliques avec plusieurs variables d'espace :

$$(*)\quad \begin{cases} \dfrac{\partial u}{\partial t} + \sum\limits_{i=1}^{N} B_i(t,x,u) \dfrac{\partial u}{\partial x_i} + C(t,x,u) = \mu \, \Delta u \ \text{ dans } \pi_T \\[2ex] u(x,o) = u_o(x) \quad \text{ dans } \mathbb{R}^N \end{cases}$$

où B_i, C et u_o sont des données, $\mu > o$ fixé et

$\pi_T =]o, T[\times \mathbb{R}^N$, $u = (u^1, u^2, \ldots, u^M)$.

On introduit une notion de solution généralisée pour (*) pour laquelle on a un résultat d'existence et d'unicité ceci à l'aide de techniques probabilistes.

Mots clefs : quasilinéaire - solution généralisée

ON GLOBAL SMOOTH SOLUTION OF CAUCHY PROBLEM FOR A CLASS OF QUASILINEAR PARABOLIC SYSTEMS IN SEVERAL SPACES VARIABLES

Omar BENNOUNA (*)

INTRODUCTION

We consider in this paper the problem of existence and uniqueness of global smooth solutions for the Cauchy problem

$$(*) \quad \begin{cases} \dfrac{\partial u}{\partial t} + \displaystyle\sum_{i=1}^{N} B_i(t,x,u)\,\dfrac{\partial u}{\partial x_i} + C(t,x,u) = \mu\Delta u & \text{in } \Pi_T \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

where B_i, C and u_0 are given data, $\mu > 0$ fixed, and $\Pi_T =]0,T[\times \mathbb{R}^N$,

$$u = (u^1, u^2, \ldots, u^M).$$

Using probabilistic methods we introduce a notion of generalized solution of (*). When this generalized solution is smooth enough then it is a classical solution. In the proof of the existence result we use a successive approximation process.

The paper is organized as follows : the part I contains definitions and some usefull lemmas, and the part II is concerned with the existence and uniqueness results.

This paper generalizes the result of D.W. STROOCK [2] on the probabilistic representation of the solution of Cauchy problem for a class of linear parabolic systems.

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I. DEFINITIONS AND PRELIMINARY RESULTS

I.1. Notations and assumptions

In the following we assume that :

(1) $B_i^{k\ell}(t,x,v)$, $C^{k\ell}(t,x,v)$ are bounded and have bounded derivatives with respect to v

(2) $u_0^k(x)$ is a bounded measurable function.

Let us denote

(3) $$G(t,x,v) = \int_0^1 \frac{\partial C}{\partial u}(t,x,\theta v)d\theta \quad , \quad \forall\ (t,x,v) \in \Pi_T \times \mathbb{R}^M$$

then

$$C(t,x,v) = G(t,x,v)v + C(t,x,0) \ .$$

In the sequel, for convenience, we take

$$C(t,x,0) \equiv 0 \ .$$

Let $\Omega = C([0,\infty),\mathbb{R}^N)$. Given $t \geq 0$ and $\omega \in \Omega$ denote by $y(t) \equiv y(t,\omega)$ the value of ω at t. For $0 \leq s \leq t$, define

$$F_t^s = \mathcal{B}[y(\lambda) : s \leq \lambda \leq t]$$

$$F^s = \mathcal{B}[y(\lambda) : \lambda \geq s] \ .$$

Then it was shown in [1] that for each $s \geq 0$ and $x \in \mathbb{R}^N$ there is a unique probability measure $P_{s,x}$ on (Ω,F^s) such that

(4) $$y(t) = x + \sqrt{2\mu}\,[w(t)-w(s)] \qquad \text{a.s. } P_{s,x}$$

where $w(t)$ is the standard Wiener process.

Let $u(t,x)$ be a bounded measurable vector function and denote by $X_u^s(t)$, $t \geq s$ the solution to the stochastic integral equation

(5) $$X_u^s(t) = I + \frac{1}{\sqrt{2\mu}} \int_s^t X_u^s(\lambda) B_i(\lambda, y(\lambda), u(\lambda, y(\lambda))) dw_i(\lambda) + \int_s^t X_u^s(\lambda) G(\lambda, y(\lambda), u(\lambda, y(\lambda))) d\lambda \qquad \text{a.s. } P_{s,x} .$$

For convenience we consider problem (*) with final condition.

I.2. Definitions and lemmas

We now give the definition of a generalized solution.

Definition

A bounded measurable vector function $u(t,x)$ is a generalized solution of (*) if $u(t,x)$ satisfies the integral equation :

(6) $$u(t,x) = E_{t,x}\, X_u^t(T) u_0(y(T)) \quad , \quad \forall\, (t,x) \in \Pi_T .$$

Remark 1

From the weak continuity of $P_{s,x}$ we have that $u \in C(\Pi_T)$.

Remark 2

When $u \in C_b^{2,1}(\Pi_T)$, using the Itô formula we see that u is a classical solution.

Let us give some usefull results :

Lemma 1 : Under conditions (1), (2), we have the estimate

(7) $$||u(t)||_\infty^2 \leq M\, ||u_0||_\infty^2 \exp\left[\frac{(T-t)}{2\mu} \left(||B||_\infty^2 + 4\mu ||G||_\infty\right)\right]$$

$\forall\, t \in [0,T]$.

Proof : We have

(8) $$|u(t,x)| \leq ||u_0||_\infty (E_{t,x}\, ||X_u^t(T)||^2)^{1/2} .$$

On the other hand we see that

(9) $$||X_u^t(T)||^2 = M + \frac{1}{\sqrt{2\mu}} \sum_i \int_t^T Tr[X_u^t(s)B_i(s,y(s),u(s,y(s)))X_u^t(s)^*]dw_i(s) +$$
$$+ \frac{1}{2\mu} \sum_i \int_t^T Tr[X_u^t(s)\; B_i(s,y(s),u(s,y(s)))$$
$$B_i^*(s,y(s),u(s,y(s)))X_u^t(s)^*]ds +$$
$$+ 2\int_t^T Tr[X_u^t(s)G(s,y(s),u(s,y(s)))X_u^t(s)^*]ds .$$

Then, using condition (3) we get that

(10) $$E||X_u^t(T)||^2 \leq M + [\frac{1}{2\mu}\, ||B||_\infty^2 + 4\mu||G||_\infty] \int_t^T E||X_u^t(s)||^2 ds$$

which implies

$$E||X_u^t(T)||^2 \leq M \exp [\frac{(T-t)}{2\mu}\, (||B||_\infty^2 + 4\mu||G||_\infty)]$$

and then (7).

Remark 3

If we assume that : $Tr[AG(t,x,v)A^*] \leq -\alpha||A||^2$ for any $(t,x,v) \in \Pi_T \times \mathbb{R}^M$ and any $M\times M$ real valued matrix A we obtain

(11) $$E||X_u^t(s)||^2 \leq M \exp [\frac{(s-t)}{2\mu}\, (||B||_\infty^2 - 4\alpha\mu)] \qquad \forall\, s,\; 0 \leq t \leq s \leq T .$$

We assume in the following that

$$(12) \qquad ||G(t,x,v_1)-G(t,x,v_2)|| \le K||v_1-v_2|| \quad \forall\ (t,x) \in \Pi_T .$$

<u>Lemma 2</u> : Under conditions (1), (2) and (12) we have

$$(13) \qquad E||X_u^t(s) - X_v^t(s)||^2 \le M_0(s-t) \int_t^s ||u(\theta)-v(\theta)||_\infty^2 \, d\theta$$

where

$$(14) \qquad M_0(s-t) = 4M.\exp[\frac{4}{\mu}(s-t)||B||_\infty^2]\exp[4(s-t)||G||_\infty(1+(s-t)||G||_\infty)]$$

$$.[||\frac{\partial G}{\partial u}||_\infty^2(s-t) + \frac{1}{2\mu}||\frac{\partial B}{\partial u}||_\infty^2]^{(*)} .$$

<u>Proof</u> :

$$(15) \qquad X_u^t(s) - X_v^t(s) = \frac{1}{\sqrt{2\mu}}\int_t^s X_u^t(\theta)[B_i(\theta,y(\theta),u(\theta,y(\theta)))-$$

$$-B_i(\theta,y(\theta),v(\theta,y(\theta)))]dw_i(\theta) +$$

$$+\frac{1}{\sqrt{2\mu}}\int_t^s [X_u^t(\theta)-X_v^t(\theta)]B_i(\theta,y(\theta),v(\theta,y(\theta)))dw_i(\theta) +$$

$$+\int_t^s X_u^t(\theta)[G(\theta,y(\theta),u(\theta,y(\theta)))-G(\theta,y(\theta),v(\theta,y(\theta)))]d\theta$$

$$+\int_t^s [X_u^t(\theta)-X_v^t(\theta)]G(\theta,y(\theta),v(\theta,y(\theta)))d\theta$$

$$= I + II + III + IV .$$

We have

(*) $||\frac{\partial f}{\partial u}||_\infty$ is the smallest Lipschitz constant.

$$(16)\qquad E(||I||^2) \le \frac{1}{2\mu} E \int_t^s ||X_u^t(\theta)[B_i(u(\theta,y(\theta))) - B_i(v(\theta,y(\theta)))]||^2 d\theta$$

$$\le \frac{1}{2\mu} ||\frac{\partial B}{\partial u}||_\infty^2 \sup_{t\le\theta\le s} (E||X_u^t(\theta)||^2) \int_t^s ||u(\theta)-v(\theta)||^2 d\theta$$

$$(17)\qquad E(||II||^2) \le \frac{1}{2\mu} E \int_t^s ||[X_u^t(\theta)-X_v^t(\theta)]B_i(v(\theta,y(\theta)))||^2 d\theta$$

$$\le \frac{1}{2\mu} ||B||_\infty^2 \int_t^s E||X_u^t(\theta)-X_v^t(\theta)||^2 d\theta$$

$$(18)\qquad E(||III||^2) \le ||\frac{\partial G}{\partial u}||_\infty^2 \sup_{t\le\theta\le s} (E||X_u^t(\theta)||^2) \int_t^s ||u(\theta)-v(\theta)||^2 d\theta (s-t)$$

$$(19)\qquad E(||IV||^2) \le ||G||_\infty^2 \int_t^s E||X_u^t(\theta)-X_v^t(\theta)||^2 d\theta (s-t) .$$

Hence, we get that

$$(20)\qquad E||X_u^t(s)-X_v^t(s)||^2 \le 4(\frac{1}{2\mu}||B||_\infty^2+||G||_\infty^2(s-t)) \int_t^s E||X_u^t(\theta)-X_v^t(\theta)||^2 d\theta +$$

$$+ 4 \sup_{t\le\theta\le s} E(||X_u^t(\theta)||^2)[||\frac{\partial G}{\partial u}||_\infty^2(s-t) + \frac{1}{2\mu}||\frac{\partial B}{\partial u}||_\infty^2]$$

$$\int_t^s ||u(\theta)-v(\theta)||_\infty^2 d\theta$$

and then, using (11) and the Gronwall lemma,

$$(21)\qquad E||X_u^t(s)-X_v^t(s)||^2 \le 4M(\exp[2(s-t)(\frac{1}{\mu}||B||_\infty^2+2||G||_\infty)])$$

$$[||\frac{\partial G}{\partial u}||^2(s-t) + \frac{1}{2\mu}||\frac{\partial B}{\partial u}||_\infty^2]$$

$$(\exp[4(s-t)(\frac{1}{2\mu}||B||_\infty^2+||G||_\infty^2(s-t)])$$

$$\int_t^s ||u(\theta)-v(\theta)||^2 d\theta$$

which completes the proof.

II. EXISTENCE AND UNIQUENESS RESULTS

Let us consider the following process of successive approximations :

(22)
$$u^0(t,x) = u_0(x)$$
$$u^n(t,x) = E_{t,x}\, X^t_{u^{n-1}}(T)u_0(y(T)) \qquad , \forall\, n,\ n \geq 1$$

where $X^t_{u^{n-1}}(s)$ is the solution of

(23)
$$X^t_{u^{n-1}}(s) = I + \frac{1}{\sqrt{2\mu}} \int_t^s X^t_{u^{n-1}}(\theta)B_i(\theta,y(\theta),u^{n-1}(\theta,y(\theta)))dw_i(\theta) + \int_t^s X^t_{u^{n-1}}(\theta)\ G(\theta,\ y(\theta),\ u^{n-1}(\theta,y(\theta)))\ d\theta$$

then, we have

__Theorem 1__ : Under conditions (1), (2) and (12) we have

(24)
$$\sup_{t\in[0,T]} ||u^{n+1}(t)-u^n(t)||^2_\infty \leq \frac{(M_1(T).T)^n}{n!} \sup_{t\in[0,T]} ||u^1(t)-u^0(t)||^2_\infty$$

where

(25)
$$M_1(T) = ||u_0||^2_\infty\, M_0(T)\ .$$

__Proof__ : Indeed, we have for $n \geq 1$

(26)
$$u^{n+1}(t,x)-u^n(t,x) = E_{t,x}\, X^t_{u^n}(T)u_0(y(T)) - E_{t,x}\, X^t_{u^{n-1}}(T)u_0(y(T))\ .$$

Thus

$$||u^{n+1}(t,x)-u^n(t,x)|| \leq E_{t,x}||u_0(y(T))||\ ||X^t_{u^n}(T)-X^t_{u^{n-1}}(T)||$$
$$\leq ||u_0||_\infty\, E_{t,x}||X^t_{u^n}(T)-X^t_{u^{n-1}}(T)||\ .$$

From (13), we get that

$$(27) \qquad ||u^{n+1}(t)-u^n(t)||_\infty^2 \leq ||u_0||_\infty^2 M_0(T-t) \int_t^T ||u^n(\theta)-u^{n-1}(\theta)||_\infty^2 \, d\theta \ .$$

Hence

$$(28) \qquad \sup_{t\in[0,T]} ||u^{n+1}(t)-u^n(t)||_\infty^2 \leq \left(||u_0||_\infty^{2n}\right)\frac{(M_0(T).T)^n}{n!} \sup_{t\in[0,T]} ||u^1(t)-u^0(t)||_\infty^2$$

which completes the proof.

Now from (24) we have the uniform convergence of the sequence $u^n(t,x)$ on $\Pi_{T,R}$ (*) to a vector valued function $u(t,x)$.

On the other hand, there is a matrix-valued function $X_u^s(t)$ such that

$$(29) \qquad X_u^s(t) = I + \frac{1}{\sqrt{2\mu}} \int_s^t X_u^s(\theta)B_i(\theta,y(\theta),u(\theta,y(\theta)))dw_i(\theta) +$$

$$+ \int_s^t X_u^s(\theta)G(\theta,y(\theta),u(\theta,y(\theta)))d\theta \qquad , \quad \text{a.s. } P_{s,x}$$

and $\mu(t,x)$ satisfies

$$(30) \qquad u(t,x) = E_{t,x}\, X_u^t(T)u_0(y(T)) \ .$$

Thus $u(t,x)$ is a generalized solution of (*).

To obtain the uniqueness result, let $(X_v^s(t),v(t,x))$ be another generalized solution of (*).

We have

$$(31) \qquad u(t,x) - v(t,x) = E_{t,x}\, X_u^t(T)u_0(y(T)) - E_{t,x}\, X_v^t(T)u_0(y(T))$$

(*) $\Pi_{T,R} =]0,T[\times B(0,R)$.

hence,

$$||u(t)-v(t)||_\infty^2 \le ||u_0||_\infty^2 \, E_{t,x} \, ||X_u^t(T)-X_v^t(T)||^2 \quad .$$

From this and (13) we obtain that

$$(32) \qquad ||u(t)-v(t)||_\infty^2 \le ||u_0||^2 \, M_0(T) \, . \int_t^T ||u(\theta)-v(\theta)||^2 d\theta \; .$$

Thus, by Gronwall's inequality we have that

$$(33) \qquad ||u(t)-v(t)||_\infty^2 \equiv 0 \quad , \quad \forall \; t \in [0,T]$$

i.e. $\quad u(t,x) = v(t,x) \quad \text{in} \quad \bar{\Pi}_T$.

Remark 4

When $B_i^{k\ell}(t,x,v)$, $G(t,x,v)$ and $u_0(x)$ have two continuous derivatives in x and v then we can obtain a priori estimate of derivatives of the generalized solution.

Remark 5

The results of this paper remain valid for the more general parabolic systems with Cauchy condition :

$$\frac{\partial u^k}{\partial t} + \sum_{i=1}^{N} \sum_{\ell=1}^{M} B_i^{k\ell}(t,x,u) \frac{\partial u^\ell}{\partial x_i} + C^k(t,x,v) = \sum_{i,j=1}^{N} a_{ij}(t,x) \frac{\partial^2 u^k}{\partial x_i \partial x_j} \quad \text{in} \quad \Pi_T$$

$$u^k(0,x) = u_0^k(x) \quad \text{in} \quad \mathbb{R}^N \quad , \quad 1 \le k \le M \; .$$

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Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique